Math in Nature:
Finding order in chaos

Jessica Kunke
Lei Zhu
Stephen Portillo
Seminar Outline

Math models and nonlinearity
Jessica

Butterfly effect and weather forecasting
Lei

Structure in chaos
Stephen
Part 1:
Math models and nonlinearity

Jessica Kunke
Outline

• Modeling our world

• Linear or nonlinear? That is the question...
Why model? To predict

1. Modeling our world

2. Linear / Nonlinear

http://www.weather.com/weather/hurricanecentral/
http://earthobservatory.nasa.gov/NaturalHazards/view.php?id=82341
Why model? To understand

1. Modeling our world

2. Linear / Nonlinear

~3x the area of Boston

http://www.youtube.com/watch?v=kLUzQjE8wU4
What’s in a model?

• equations (relationships)
• not exact reality, but useful
  – purposes: predict, understand
  – can’t simulate exact reality
  – to simulate ≠ to understand
• involves making simplifications
• no single model
• can use observations
• might use computers

1. Modeling our world
2. Linear / Nonlinear

http://www.weather.com/weather/hurricanecentral/
http://www.youtube.com/watch?v=kLUzQjE8wU4
An example: crop yield

light? water? temperature? soil composition?
pollution? pests? wind?
genetics? management? weeds? crop rotation?
...other?
An example: crop yield

- light
- water
- temperature
- soil composition?
- pollution?
- pests?
- wind?
- genetics?
- management?
- weeds?
- crop rotation?
- ...other?

http://openclipart.org/detail/168724/simple-farm-crops-by-viscious-speed
http://www.50states.com/maps/massachusetts.htm#UoMhsY29zAo
1. Modeling our world

2. Linear / Nonlinear

An example: crop yield

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- water
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- pests
- pollution?
- wind?
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- crop rotation?
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An example: crop yield

- light
- water
- temperature
- soil composition
- pests
- color
- duration

Wind?
Weeds?
Crop rotation?
...other?

http://openclipart.org/detail/168724/simple-farm-crops-by-viscious-speed
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What’s in a model?

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✓ might use computers
Questions?
Ed Lorenz and the Discovery of Chaos, 1961

Value of Weather Variable vs Time

0.506127 vs 0.506

error of about 3 in 10,000!

http://www.ucar.edu/communications/quarterly/spring08/ed_lorenz.jsp
http://mcherm.com/permalinks/1/the-butterfly-effect
1. Modeling our world

2. Linear / Nonlinear

A Linear Model

Independent

Interacting

Time

http://openclipart.org/detail/22287/cartoon-zebra-by-studiofibonacci
http://openclipart.org/detail/36937/gazelle-by-papapishu
1. Modeling our world

A **Non**linear Model

2. Linear / Nonlinear

<table>
<thead>
<tr>
<th>Independent</th>
<th>Interacting</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="http://openclipart.org/detail/22287/cartoon-zebra-by-studiofibonacci" alt="Lion images" /> + <img src="http://openclipart.org/detail/31975/architetto----leone-02-by-anonymous" alt="Zebra images" /></td>
<td><img src="http://openclipart.org/detail/22287/cartoon-zebra-by-studiofibonacci" alt="Question mark" /> ≠</td>
</tr>
</tbody>
</table>
1. Modeling our world

A **Non**linear Model

2. Linear / Nonlinear

**Independent**

**Interacting**

Time

http://openclipart.org/detail/22287/cartoon-zebra-by-studiofibonacci
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1. Modeling our world

A **Non**linear Model

2. Linear / Nonlinear

- Independent
- Interacting

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1. Modeling our world

2. Linear / Nonlinear

A Nonlinear Model

Independent

Interacting

No superposition!

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1. Modeling our world

A **Non**linear Model

- Independent

- Interacting

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1. Modeling our world

A **Nonlinear** Model

- Independent
- Interacting

Time

Sensitivity to initial conditions

http://openclipart.org/detail/22287/cartoon-zebra-by-studiofibonacci
http://openclipart.org/detail/31975/architetto----leone-02-by-anonymous
# Summary: Linear vs Non-linear

<table>
<thead>
<tr>
<th>Superposition</th>
<th>Linear</th>
<th>Non-linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>– The result of the combination is the same as the combination of the separate results</td>
<td>Yes!</td>
<td>Nope</td>
</tr>
</tbody>
</table>
Modeling our world

...nonlinearity is everywhere!
Summary

• Models help us study nature

• Nonlinear models
  – Superposition
  – Sensitivity to initial conditions
  – A note about feedbacks and interactions

• Nonlinearity is everywhere!
Part II
Butterfly effect and weather forecasting

Lei Zhu
Phd student
Atmospheric chemistry
Butterfly effect in popular culture

When a butterfly flaps its wings in one part of the world it can cause a hurricane in another part of the world.
A metaphor from Edward Lorenz

One meteorologist remarked that if the theory were correct, one flap of a seagull’s wings would be enough to alter the course of the weather forever.

-Lorenz, 1963

Predictability: Does the Flap of a Butterfly’s Wings in Brazil set off a Tornado in Texas?

-Lorenz, 1972

Seagull evolved into the more poetic butterfly
“Nail effect”: An old poem

Lessons we learn:

A tiny change can lead to a huge difference in the final results

For want of a nail the shoe was lost,
for want of a shoe the horse was lost;
and for want of a horse the rider was lost;
being overtaken and slain by the enemy,
all for want of care about a horse-shoe nail.

-Benjamin Franklin
Butterfly effect: from the perspective of chaos theory

- Sensitivity to the initial conditions
- Every observation has errors
- The error will be propagated through the nonlinear system
- Thus, it’s almost impossible to make long time predictions in a nonlinear system
Butterfly effect: An example

Consider a temperature forecasting model:

\[ \begin{cases} Y_t = 1.5 \times Y_{t-1}^2 - 1 \\ T_t = 30 \times (Y_t + 1) + 60 \end{cases} \]

Initial condition from observation
Butterfly effect: Any pattern?

Can you see any pattern in simulations with different initial conditions?

Consider our temperature model again:
Weather forecasting

• Short term forecasting (~12h) is very accurate
• Long term forecasting is impossible.
• But we can predict the probability!
Butterfly effect: Predict the probability

Run the simulation for 1000 times with slightly different initial conditions.
Butterfly effect: Predict the probability

Time series for 1000 simulations

60% confidence

600 of the 1000 simulations: Range from 60.9-68.3 F
Weather forecasting

• What improves our prediction capabilities?

Forecasting accuracy from 1981 to 2011
Summary

• Butterfly effect is nothing more than a metaphor
• Long term weather forecasting is impossible
• But we can predict the probability
• There is a general pattern/rule/structure in a chaotic system
Part III:
Structure in Chaos

Stephen Portillo
Understanding the Patterns

It’s impossible to predict the exact behaviour of non-linear systems,

But non-linear systems have patterns.

(a) T1, Initial=100

(b) T2, Initial=100+1

(c) T2-T1

What can we learn about these patterns?
Outline

• The Logistic Equation
• Non-Linear Math and Self-Similarity
• Self-Similarity in Nature
AN EXAMPLE:
THE LOGISTIC EQUATION
Simplify the Math

To predict, we need a complicated model

To understand, we start with a simple model

Joint Typhoon Warning Center
www.usno.navy.mil/JTWC/
European Centre for Medium-Range Weather Forecasts
www.ecmwf.int

National Weather Service
http://www.srh.noaa.gov/jetstream/tropics/tc_structure.htm
A First Ecological Model

- Consider an animal population that grows each year
  \[ P_{next} = rP \]

- \( P = \text{population this year} \)
- \( P_{next} = \text{population next year} \)
- \( r = \text{reproductive rate} \)

**Year 1**
- Population 100

**Year 2**
- Population 200

**Year 3**
- Population 400

\( r = 2 \)
A First Ecological Model

• Consider an animal population that grows each year

\[ P_{next} = rP \]

• \( P = \text{population this year} \)
• \( P_{next} = \text{population next year} \)
• \( r = \text{reproductive rate} \)
The Logistic Equation

• Take the logistic equation

$$P_{\text{next}} = rP(1-P/C)$$

• $C =$ carrying capacity, the population where all resources would be exhausted

$\begin{align*}
\text{Year 1} & : \text{Population 100} \\
\text{Year 2} & : \text{Population 180} \\
\text{Year 3} & : \text{Population 295}
\end{align*}$

$r=2, C=1000$
The Logistic Equation

- Take the logistic equation

\[ P_{next} = rP(1 - P/C) \]

- \( C = \text{carrying capacity}, \) the population where all resources would be exhausted

\[ r=2, \ C=1000 \]

Year 1
- Population 100

Year 2
- Population 180
  - Population 200

Year 3
- Population 295
  - Population 320

Year 3
- Population 435
  - Population 435

Non-linear!
Demo

- Carrying capacity of 1,000,000
- Reproductive rate of 2.5
Meet the Bifurcation Diagram

Even a simple equation can give complex behaviour – if it’s non-linear.

\[ P_{n+1} = rP(1-P/C) \]
Questions?
NON-LINEAR MATH AND SELF-SIMILARITY
Self-Similarity
Feigenbaum’s Constant

$\delta = 4.669201609...$

$4.233...$
Universality

- Is this pattern just for the logistic equation?

\[ P_{\text{next}} = rP(1 - P/C) \]

- Now for something completely different...

\[ P_{\text{next}} = r\sin\pi P \]
Universality

This self-similarity arises from the math itself.
SELF-SIMILARITY IN NATURE
Self-Similarity in Nature

Wikimedia Commons (user:Olegiwit)
Self-Similarity in Nature

Fractal Geometry, Yale University
http://classes.yale.edu/fractals/
Self-Similarity in Nature

Wikimedia Commons (user:Yann)
Self-Similarity in Nature

Wikimedia Commons (terraprints.com)
Self-Similarity in Nature

Smithsonian Institute
Self-Similarity in Nature

The Millenium Simulation
http://www.mpa-garching.mpg.de/galform/millennium/
The Nature of Self-Similarity

• Nature is described by non-linear math
• Non-linear math gives rise to self-similarity
• Nature has self-similarity arising from non-linear math
Conclusion

• We understand nature by using mathematical models, often non-linear
• We can’t calculate the exact behaviour of non-linear models, but we can still make useful predictions
• Self-similarity arises from non-linear math itself, and is related to self-similarity in nature
Thank you!

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The Harvard Biomedical Graduate Students Organization (BGSO)

The Harvard/MIT COOP

Restaurant Associates
1. Modeling our world

2. Linear / Nonlinear

Linear Systems
Why model?  To understand

1. Modeling our world

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http://www.youtube.com/watch?v=kLUzQjE8wU4
http://www.youtube.com/watch?v=chIzYtJjxhc